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Free Oscillations of a Gravitating Solid Sphere

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CONTENTS

ABSTRACT

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The problem of determining the free oscillations of a gravitating solid sphere is investigated. The problem itself is formulated and, with the assumption of radial symmetry in the structure of the sphere, the numerical solution for the case of the toroidal oscillations is described in detail. A formal description is given to indicate the numerical solution of the problem of the spheroidal oscillations.

I. INTRODUCTION

There has been considerable recent interest in the theoretical determination of the free oscillations of the Earth. Seismic data obtained from the Chilean earthquake of 1960, as well as from the Kamchatka earthquake of 1952, have been analyzed statistically to obtain an experimental determination of the free oscillations of the Earth (Ref. 1). Correlation of these results with the theoretically calculated free

oscillations corresponding to a number of hypothetical Earth models has afforded valuable information regarding the probable internal structure of the Earth.

It is to be expected that many of the seismic techniques which have proved valuable for investigating the Earth can, with some modification, be applied fruitfully to the investigation of the corresponding lunar and planetary phenomena. It may be of particular interest to ascertain whether any of these bodies possess a solid core. As a first step in such an investigation, it appears desirable to be able to determine the free oscillations of a gravitating solid sphere.

Within the restriction that the assumed structure of the sphere has radial symmetry, there has been prepared for the Jet Propulsion Laboratory IBM 7090 a program by which the toroidal oscillations of a gravitating solid sphere can be computed. A program for computing the spheroidal oscillations is in progress.

II. FORMULATION OF THE PROBLEM

The stress equations of small motion of any body for which the notion of stress is valid are given by Love (Ref. 2) in rectangular coordinates

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \rho X = \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + \rho Y = \rho \frac{\partial^2 u_y}{\partial t^2} \qquad (1)$$

$$\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + \rho Z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

where u_x, u_y, and u_z denote the components of the displacement vector $\mathbf{u}$ in the x, y, and z directions respectively.

In the case of polar coordinates r, θ, and ϕ, the corresponding stress equations are (see Ref. 2, p. 91)

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r} \frac{\partial \widehat{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \widehat{r\phi}}{\partial \phi} + \frac{1}{r} (2 \widehat{rr} - \widehat{\theta\theta} - \widehat{\phi\phi} + \widehat{r\theta} \cot \theta) + \rho F_r = \frac{\partial^2 u_r}{\partial t^2}$$

$$\frac{\partial \widehat{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \widehat{\theta\phi}}{\partial \phi} + \frac{1}{r} [(\widehat{\theta\theta} - \widehat{\phi\phi}) \cot \theta + 3 \widehat{r\theta}] + \rho F_\theta = \frac{\partial^2 u_\theta}{\partial t^2} \qquad (2)$$

$$\frac{\partial \widehat{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \widehat{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \widehat{\phi\phi}}{\partial \phi} + \frac{1}{r} (3 \widehat{r\phi} + 2 \widehat{\theta\phi} \cot \theta) + \rho F_\phi = \frac{\partial^2 u_\phi}{\partial t^2}$$

where u_r, u_θ, and u_ϕ denote the components of the displacement vector $\mathbf{u}$ in the r, θ, and ϕ directions respectively.

In order to proceed further, some sort of stress—strain relationship must be postulated for the body. Perhaps the simplest assumption is that the body is an isotropic elastic solid and that one may use Hooke's law

$$X_x = \lambda \Delta + 2\mu \frac{\partial u_x}{\partial x} \ , \quad Y_x = X_y = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

$$Y_y = \lambda \Delta + 2\mu \frac{\partial u_y}{\partial y} \ , \quad Y_z = Z_y = \mu \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \tag{3}$$

$$Z_z = \lambda \Delta + 2\mu \frac{\partial u_z}{\partial z} \ , \quad Z_x = X_z = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)$$

where

$$\Delta \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \equiv \nabla \cdot \mathbf{u} \tag{4}$$

If the medium is further assumed to be heterogeneous, so that λ, μ, and ρ are functions of position, then substitution of Eq. (3) and (4) into (1) gives a result which can be expressed in the vector form

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho \mathbf{F} + \nabla \left[(\lambda + \mu) \nabla \cdot \mathbf{u} \right] + \mu \nabla^2 \mathbf{u} - (\nabla \cdot \mathbf{u}) \nabla \mu + 2 (\nabla \mu \cdot \overline{\overline{\Phi}}) \tag{5}$$

where

$$\mathbf{F} \equiv \mathbf{i} X + \mathbf{j} Y + \mathbf{k} Z \tag{6}$$

and $\overline{\overline{\Phi}}$ is the symmetrical strain tensor with components

$$\frac{\partial u_x}{\partial x} \qquad \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad \frac{\partial u_y}{\partial y} \qquad \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \qquad \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \qquad \frac{\partial u_z}{\partial z}$$

If the isotropic elastic medium is assumed to be homogeneous, Eq. (5) becomes

$$\rho\,\frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho\,\mathbf{F} + (\lambda + \mu)\,\nabla\,(\nabla \cdot \mathbf{u}) + \mu\,\nabla^2\,\mathbf{u} \tag{7}$$

In either Eq. (5) or (7), any effect of gravity must be accounted for in some manner in the body forces **F**.

One approach to take in the treatment of the solid sphere with radial symmetry is to consider that the sphere is made up of concentric spherical shells, each shell assumed to be homogeneous. Assigning values to ρ, λ, and μ in each shell, we then require that Eq. (7) be satisfied in each shell, that there be no stress at the outer surface of the outer shell, and that stress components acting through the boundary and displacements be continuous at the interfaces between shells. In the case of the toroidal oscillations there is no radial displacement of the boundaries and, effectively, one may set the **F** in Eq. (7) equal to zero. The problem may then be approached using the theory of spherical harmonics (Ref. 3). The author first attacked the case of the free toroidal oscillations of a radially symmetric solid sphere in this manner and obtained a formal solution analogous to that obtained for the Earth (liquid core) by Gilbert and MacDonald (Ref. 4). This solution is not documented in the present paper since the extension of their method to the case of the spheroidal oscillations is not at all clear.

A more natural way to bring in the effect of gravity is to include it in the stress – strain assumptions. A method, devised by Lord Rayleigh (Ref. 5) and well elucidated by Love (Ref. 6), views the sphere as being in a state of initial stress which is regarded as a hydrostatic pressure balancing the self-gravitation of the body in the initial state; any subsequent displacement is considered to conform to the stress – strain relations

$$X_x = -p_0 + u_r \frac{\partial p_0}{\partial r} + \lambda \Delta + 2\mu \frac{\partial u_x}{\partial x}$$

$$Y_y = -p_0 + u_r \frac{\partial p_0}{\partial r} + \lambda \Delta + 2\mu \frac{\partial u_y}{\partial y}$$

$$Z_z = -p_0 + u_r \frac{\partial p_0}{\partial r} + \lambda \Delta + 2\mu \frac{\partial u_z}{\partial z}$$

$$X_y = Y_x = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

$$Y_z = Z_y = \mu \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right)$$

$$Z_x = X_z = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)$$

(8)

where p_0 is the initial pressure.

Expressed in polar coordinates, the stress–strain relations (8) become

$$\widehat{rr} = -p_0 + u_r \frac{\partial p_0}{\partial r} + \lambda \Delta + 2\mu\, e_{rr}$$

$$\widehat{\theta\theta} = -p_0 + u_r \frac{\partial p_0}{\partial r} + \lambda \Delta + 2\mu\, e_{\theta\theta}$$

$$\widehat{\phi\phi} = -p_0 + u_r \frac{\partial p_0}{\partial r} + \lambda \Delta + 2\mu\, e_{\phi\phi}$$

$$\widehat{r\theta} = \mu\, e_{r\theta}$$

$$\widehat{r\phi} = \mu\, e_{r\phi}$$

$$\widehat{\theta\phi} = \mu\, e_{\theta\phi}$$

(8a)

where

$$e_{rr} = \frac{\partial u_r}{\partial r}$$

$$e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$e_{\phi\phi} = \frac{1}{r \sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}$$

$$e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

$$e_{r\phi} = \frac{1}{r \sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}$$

$$e_{\theta\phi} = \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r}\cot\theta + \frac{1}{r \sin\theta}\frac{\partial u_\theta}{\partial \phi}$$

$$\Delta = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial u_\phi}{\partial \phi}$$

The body forces **F** are assumed derivable from a potential v such that

$$\mathbf{F} = \nabla v \tag{9}$$

The initial pressure p_0, the initial density ρ_0, and the initial potential v_0, all relating to the undisturbed sphere, are then assumed to be only functions of r, the distance from the center of the sphere. The quantities p_0, ρ_0, and v_0 are related by

$$\rho_0 \frac{\partial v_0}{\partial x} - \frac{\partial p_0}{\partial x} = 0$$

$$\rho_0 \frac{\partial v_0}{\partial y} - \frac{\partial p_0}{\partial y} = 0 \tag{10}$$

$$\rho_0 \frac{\partial v_0}{\partial z} - \frac{\partial p_0}{\partial z} = 0$$

or

$$\rho_0 \frac{\partial v_0}{\partial r} - \frac{\partial p_0}{\partial r} = 0 \tag{10a}$$

In the strained state, the density is assumed given by

$$\rho = \rho_0 - u_r \frac{\partial \rho_0}{\partial r} - \rho_0 \, \Delta \tag{11}$$

and the potential v given by

$$v = v_0 + W \tag{12}$$

where

$$\nabla^2 v_0 = - 4 \pi G \rho_0 \tag{13}$$

$$\nabla^2 W = 4 \pi G \left(\rho_0 \, \Delta + u_r \frac{\partial \rho_0}{\partial r} \right) \tag{14}$$

Note that if we write

$$g_0 = - \frac{\partial v_0}{\partial r} \tag{15}$$

we can replace Eq. (10a) by

$$\frac{\partial p_0}{\partial r} = - g_0 \, \rho_0 \tag{10b}$$

and Eq. (13) by

$$\frac{dg_0}{dr} + \frac{2}{r} \, g_0 = 4 \, \pi \, G \, \rho_0 \tag{13a}$$

Substituting Eq. (8) into Eq. (1), making use of (10), (10a), (11), and (12), and neglecting the products of small quantities of the first order in u_x, u_y, u_z, $\triangle$, W, and their derivatives, gives the following result expressed in vector form:

$$\rho_0 \, \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla \, [(\lambda + \mu) \, \nabla \cdot \mathbf{u}] + \mu \, \nabla^2 \, \mathbf{u} - (\nabla \cdot \mathbf{u}) \, \nabla \mu + 2 \, (\nabla \mu \cdot \overline{\overline{\Phi}})$$

$$- \left(\rho_0 \, \nabla \cdot \mathbf{u} + u_r \, \frac{\partial \rho_0}{\partial r} \right) \, \nabla \, v_0 + \rho_0 \, \nabla \, W + \nabla \, \left(u_r \, \rho_0 \, \frac{\partial v_0}{\partial r} \right) \tag{16}$$

If the medium is assumed to have λ, μ, and ρ_0 constant, then Eq. (16) becomes

$$\rho_0 \, \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \, \nabla \, (\nabla \cdot \mathbf{u}) + \mu \, \nabla^2 \, \mu - \rho_0 \, (\nabla \cdot \mathbf{u}) \, \nabla \, v_0 + \rho_0 \, \nabla \, W + \rho_0 \, \left(u_r \, \frac{\partial v_0}{\partial r} \right) \tag{17}$$

(It may be interesting to compare Eq. (5) and (7) with Eq. (16) and (17), respectively)

Of particular interest are the three scalar equations in polar coordinates corresponding to vector equation (16). These scalar equations may be obtained by substituting Eq. (8a), (9), (10a), and (15) into Eq. (2):

$$\rho_0 \frac{\partial^2 u_r}{\partial t^2} = \rho_0 g_0 \Delta + \rho_0 \frac{\partial W}{\partial r} - \rho_0 \frac{\partial}{\partial r} (g_0 u_r) + \frac{\partial}{\partial r} \left(\lambda \Delta + 2\mu \frac{\partial u_r}{\partial r} \right)$$

$$+ \frac{1}{r} \frac{\partial}{\partial \theta} (\mu e_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\mu e_{r\phi}) + \frac{\mu}{r} (4 e_{rr} - 2 e_{\theta\theta} - 2 e_{\phi\phi} + \cot \theta e_{r\theta})$$

$$(18)$$

$$\rho_0 \frac{\partial^2 u_\theta}{\partial t^2} = \frac{\rho_0}{r} \frac{\partial W}{\partial \theta} + \frac{\partial}{\partial r} (u e_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta} (-g_0 \rho_0 u_r + \lambda \Delta + 2\mu e_{\theta\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\mu e_{\theta\phi})$$

$$+ \frac{\mu}{r} \left[2 \cot \theta \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r} \cot \theta - \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right) + 3 e_{r\theta} \right]$$

$$(19)$$

$$\rho_0 \frac{\partial^2 u_\phi}{\partial t^2} = \frac{\rho_0}{r \sin \theta} \frac{\partial W}{\partial \phi} + \frac{\partial}{\partial r} (\mu e_{r\phi}) + \frac{1}{r} \frac{\partial}{\partial \theta} (\mu e_{\theta\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (-g_0 \rho_0 u_r + \lambda \Delta + 2\mu e_{\phi\phi})$$

$$+ \frac{3\mu}{r} e_{r\phi} + \frac{2\mu}{r} \cot \theta e_{\theta\phi}$$

$$(20)$$

The system of Eq. (13a), (14), (18), (19), and (20) is to be solved subject to the following boundary conditions:

(i) $\widehat{rr} = \widehat{r\theta} = \widehat{r\phi} = 0$ at $r = a + u_r$

   where a is the radius of the sphere before deformation;

(ii) The solution is to be regular at the center of the sphere;

(iii) The sphere's internal and external gravitational potentials and also their respective gradients are continuous at $r = a + u_r$.

If only the first-order terms in the small quantities u_r, u_θ, and u_ϕ are retained, the boundary conditions (i) are met by putting

$$\lambda \Delta + 2\mu \frac{\partial u_r}{\partial r} = 0 \qquad \text{at } r = a \qquad (21)$$

$$e_{r\theta} = e_{r\phi} = 0 \qquad \text{at } r = a \qquad (22)$$

The boundary conditions (iii) are similarly met by

$$W = W_s, \quad \frac{\partial W}{\partial r} - \frac{\partial W_s}{\partial r} = 4\pi G \rho_0 u_r \qquad \text{at } r = a \qquad (23)$$

The quantity W_s, which denotes the gravitational potential outside the sphere, can be expressed by

$$W_s = \sum_n a_n W_{sn} = \sum_n a_n \frac{S_n(\theta,\phi)}{r^{n+1}}$$

where $S_n(\theta,\phi)$ is a spherical surface harmonic of positive integral degree n. Since each spherical harmonic component W_{sn} satisfies

$$\frac{\partial W_{sn}}{\partial r} = -\frac{n+1}{r} W_{sn}$$

then Eq. (23) can be replaced by

$$\frac{\partial W_n}{\partial r} + \frac{n+1}{a} W_n = 4\pi G \rho_0 u_{rn} \qquad (23a)$$

where W_n and u_{rn} are the nth spherical harmonic components of W and u_r respectively.

[Boundary conditions (ii) will be discussed subsequently.]

The remainder of our development will assume a radially symmetric sphere and will be confined to two types of solutions suggested by the work of Hoskins (Ref. 7). In one type of solution, leading to the free toroidal oscillations, we assume

$$u_r = 0$$

$$u_\theta = \frac{v(r)}{\sin \theta} \frac{\partial S_n(\theta, \phi)}{\partial \phi} e^{i\sigma t} \tag{24}$$

$$u_\phi = - \frac{v(r)}{\sin \theta} \frac{\partial S_n(\theta, \phi)}{\partial \theta} e^{i\sigma t}$$

Note that Eq. (24) gives

$$\Delta = 0 \tag{25}$$

Also, with the density remaining constant and no radial displacement of the boundaries,

$$W = 0 \tag{26}$$

Substituting Eq. (24), (25), and (26) into Eq. (14), (18), (19), and (20), we find that Eq. (14) and (18) are satisfied identically while Eq. (19) and (20) both give

$$\mu \left(\frac{d^2 v}{dr^2} + \frac{2}{r} \frac{dv}{dr} \right) + \frac{d\mu}{dr} \left(\frac{dv}{dr} - \frac{v}{r} \right) + \left[\sigma^2 \rho_0 - \frac{n(n+1)\mu}{r^2} \right] v = 0 \tag{27}$$

Boundary condition (21) is satisfied identically and boundary conditions (22) both give

$$\frac{dv}{dr} - \frac{v}{r} = 0 \qquad \qquad \text{at } r = a$$

The substitutions

$$y_1 = v, \qquad\qquad y_2 = \mu \left(\frac{dv}{dr} - \frac{v}{r} \right)$$

lead to the system of equations

$$\frac{dy_1}{dr} = \frac{1}{r} y_1 + \frac{1}{\mu} y_2$$

$$\frac{dy_2}{dr} = \left[\frac{\mu (n^2 + n - 2)}{r^2} - \sigma^2 \rho_0 \right] y_1 - \frac{3}{r} y_2$$

(28)

with boundary conditions

$$y_2 = 0 \qquad\qquad \text{at } r = a \qquad\qquad (29)$$

$$y_1 \text{ is regular} \qquad\qquad \text{at } r = 0 \qquad\qquad (30)$$

In the second type of solution, leading to the free spheroidal oscillations, we assume

$$u_r = U(r) S_n (\theta, \phi) e^{i\sigma t}$$

$$u_\theta = V(r) \frac{\partial S_n (\theta, \phi)}{\partial \theta} e^{i\sigma t} \qquad\qquad (31)$$

$$u_\phi = \frac{V(r)}{\sin \theta} \frac{\partial S_n (\theta, \phi)}{\partial \phi} e^{i\sigma t}$$

We then find

$$\Delta = X(r) S_n (\theta, \phi) e^{i\sigma t} \qquad\qquad (32)$$

where

$$X = \frac{dU}{dr} + \frac{2}{r} U - \frac{n(n+1)}{r} V \tag{33}$$

We then write

$$W = P(r) S_n (\theta, \phi) e^{i\sigma t} \tag{34}$$

and, substituting Eq. (31), (32), (33), and (34) into Eq. (14), (18), (19), and (20), we obtain the following system of equations:

$$\sigma^2 \rho_0 U + \rho_0 \frac{dP}{dr} + g_0 \rho_0 X - \rho_0 \frac{d}{dr}(g_0 U) + \frac{d}{dr}\left(\lambda X + 2\mu \frac{dU}{dr}\right)$$

$$+ \frac{\mu}{r^2}\left[4r \frac{dU}{dr} - 4U + n(n+1)\left(-U - r \frac{dV}{dr} + 3V\right)\right] = 0$$

$$\rho_0 \sigma^2 Vr + \rho_0 P - g_0 \rho_0 U + \lambda X + r \frac{d}{dr}\left[\mu\left(\frac{dV}{dr} - \frac{V}{r} + \frac{U}{r}\right)\right] \tag{35}$$

$$+ \frac{\mu}{r}\left[5U + 3r \frac{dV}{dr} - V - 2n(n+1)V\right] = 0$$

$$\frac{d^2 P}{dr^2} + \frac{2}{r} \frac{dP}{dr} - \frac{n(n+1)}{r^2} P = 4\pi G \left(\frac{d\rho_0}{dr} U + \rho_0 x\right)$$

Boundary condition (21) gives

$$2X + 2\mu \frac{dU}{dr} = 0 \qquad\qquad \text{at } r = a \tag{36}$$

while boundary conditions (22) give

$$\mu \left(\frac{dV}{dr} - \frac{V}{r} + \frac{U}{r} \right) = 0 \qquad \text{at } r = a \qquad (37)$$

Boundary conditions (23) give (see Eq. 23a)

$$\frac{dP}{dr} + \frac{n+1}{a} P = 4 \pi G \rho_0 U \qquad \text{at } r = a \qquad (38)$$

The substitutions

$$y_1 = U$$

$$y_2 = \lambda X + 2\mu \frac{dU}{dr}$$

$$y_3 = V$$

$$y_4 = \mu \left(\frac{dV}{dr} - \frac{V}{r} + \frac{U}{r} \right) \qquad (39)$$

$$y_5 = P$$

$$y_6 = \frac{dP}{dr} - 4 \pi G \rho_0 U$$

lead to the system of equations

$$\frac{dy_1}{dr} = -\frac{2\lambda}{(\lambda + 2\mu)}\frac{y_1}{r} + \frac{1}{(\lambda + 2\mu)}y_2 + \frac{\lambda n(n+1)}{(\lambda + 2\mu)}\frac{y_3}{r}$$

$$\frac{dy_2}{dr} = \left[-\sigma^2 \rho_0 r^2 - 4\rho_0 g_0 r + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)}\right]\frac{y_1}{r^2} - \frac{4\mu}{(\lambda + 2\mu)}\frac{y_2}{r}$$

$$+ \left[n(n+1)\rho_0 g_0 r - \frac{2\mu(3\lambda + 2\mu)n(n+1)}{(\lambda + 2\mu)}\right]\frac{y_3}{r^2} + n(n+1)\frac{y_4}{r} - \rho_0 y_6$$

$$\frac{dy_3}{dr} = -\frac{y_1}{r} + \frac{y_3}{r} + \frac{y_4}{\mu}$$

$$\frac{dy_4}{dr} = \left[g_0 \rho_0 r - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)}\right]\frac{y_1}{r^2} - \frac{\lambda}{(\lambda + 2\mu)}\frac{y_2}{r} \qquad (40)$$

$$+ \left\{-\rho_0 \sigma^2 r^2 + \frac{2\mu}{(\lambda + 2\mu)}[\lambda(2n^2 + 2n - 1) + 2\mu(n^2 + n - 1)]\right\}\frac{y_3}{r^2} - \frac{3y_4}{r} - \frac{\rho_0 y_5}{r}$$

$$\frac{dy_5}{dr} = 4\pi G \rho_0 y_1 + y_6$$

$$\frac{dy_6}{dr} = -4\pi G \rho_0 n(n+1)\frac{y_3}{r} + n(n+1)\frac{y_5}{r^2} - \frac{2y_6}{r}$$

$$\frac{dg_0}{dr} + \frac{2}{r}g_0 = 4\pi G \rho_0$$

with boundary conditions

$$y_2 = y_4 = 0 \quad , \quad y_6 + \frac{n+1}{a} y_5 = 0 \qquad \text{at } r = a \qquad (41)$$

$$y_1, y_3, \text{ and } y_5 \text{ are regular} \qquad \text{at } r = 0 \qquad (42)$$

and

$$g_0 = 0 \qquad \text{at } r = 0 \qquad (43)$$

The equation

$$\frac{dg_0}{dr} + \frac{2}{r} g_0 = 4\pi G \rho_0$$

with boundary condition (43) can be replaced by

$$g_0 (r) = \frac{1}{r^2} \int_0^r 4\pi G \rho_0 r^2 dr \qquad \text{for } r > 0$$

$$\qquad (44)$$

$$= 0 \qquad \text{for } r = 0$$

The above formulation closely parallels that of Alterman, Jarosch, and Pekeris (Ref. 8), except that they were concerned with a sphere with a liquid core. Since there can be no toroidal oscillations in a liquid core, the condition of regularity at the origin did not have to be met in that case. The equations for the spheroidal oscillations in the liquid core are easily obtainable from Eq. (40) by setting

$$\mu = 0, \qquad y_2 = \lambda x, \qquad y_4 = 0$$

and give rise to a set of four differential equations which include among the boundary conditions the requirement that y_1 and y_5 be regular at the origin. Alterman, Jarosch, and Pekeris (Ref. 8) do not clarify the manner in which the conditions of regularity were imposed in their numerical solution.

III. NUMERICAL SOLUTION OF THE PROBLEM

The chosen method of solving the differential equations—which also satisfies the boundary conditions—can be illustrated by a consideration of the case of the toroidal oscillations.

By introducing the variable x, where

$$x = \frac{r}{a} \tag{45}$$

Eq. (28) becomes

$$\frac{dy_1}{dx} = \frac{1}{x} y_1 + \frac{a}{\mu} y_2$$

$$\frac{dy_2}{dx} = \left[\frac{\mu(n^2 + n - 2)}{ax^2} - a\sigma^2 \rho_0 \right] y_1 - \frac{3}{x} y_2 \tag{46}$$

while boundary conditions (29) and (30) become

$$y_2 = 0 \qquad\qquad \text{at } x = 1 \tag{47}$$

$$y_1 \text{ is regular} \qquad\qquad \text{at } x = 0 \tag{48}$$

The condition of regularity at the origin means that in the neighborhood of the origin

$$y_1 = x^s (A_0 + A_1 x + \cdots + A_k x^k + \cdots) \tag{49}$$

where

$$A_0 \neq 0 \text{ and } s \geq 0 \tag{50}$$

From Eq. (46), it is seen that

$$y_2 = \frac{\mu}{a} \left(\frac{dy_1}{dx} - \frac{y_1}{x} \right) \tag{51}$$

If we now assume that within a sufficiently close neighborhood of the origin μ and ρ_0 are constant, say

$$\mu = \bar{\mu}, \quad \rho_0 = \bar{\rho} \qquad\qquad \text{for } x \leq h_1 \tag{52}$$

then we find that the second equation of Eq. (46), along with conditions (50), requires that as long as $n \nless 1$,

$$s = n \tag{53}$$

$$A_{2l} = \frac{(-\bar{\rho} \, a^2 \, \sigma^2)^l}{(2\bar{\mu})^l \, l! \, (2n + 3)(2n + 5) \cdots (2n + 2l + 1)} \, A_0 \tag{54}$$

$$A_{2l+1} = 0$$

for $l = 1, 2, 3, \cdots$

Then, as long as $n \nless 1$, we have within a close neighborhood of the origin (say, for $x \leq h \leq h_1$)

$$
\begin{aligned}
y_1 = A_0 \, x^n \Bigg[& 1 - \frac{1}{2(2n+3)} \left(\frac{a^2 \, \bar{\rho} \sigma^2 x^2}{\bar{\mu}} \right) \\
& + \cdots + (-1)^l \frac{1}{2^l \, l! \, (2n+3)(2n+5)\cdots(2n+2l+1)} \left(\frac{a^2 \, \bar{\rho} \, \sigma^2 x^2}{\bar{\mu}} \right)^l + \cdots \Bigg]
\end{aligned}
$$

$$\tag{55}$$

$$
\begin{aligned}
y_2 = A_0 \, \frac{\bar{\mu}}{a} \, x^{n-1} \Bigg[& (n-1) - \frac{(n+1)}{2(2n+3)} \left(\frac{a^2 \bar{\rho} \sigma^2 x^2}{\bar{\mu}} \right) \\
& + \cdots + (-1)^l \frac{(n+2l-1)}{2^l \, l! \, (2n+3)(2n+5)\cdots(2n+2l+1)} \left(\frac{a^2 \bar{\rho} \sigma^2 x^2}{\bar{\mu}} \right)^l + \cdots \Bigg]
\end{aligned}
$$

which will be abbreviated as

$$y_1 = A_0 f_1(x)$$

(56)

$$y_2 = A_0 f_2(x)$$

for $x \leq h$.

In the case that $n = 1$ we see that the set of differential equations (46) is replaced by

$$\frac{dy_1}{dx} = \frac{1}{x} y_1 + \frac{a}{\mu} y_2$$

(46a)

$$\frac{dy_2}{dx} = - a \sigma^2 \rho_0 y_1 - \frac{3}{x} y_2$$

In this case we obtain

$$s = 1$$

$$A_{2l} = \frac{(- \bar{\rho} a^2 \sigma^2)^l}{(2\bar{\mu})^l \, l! \, (5)(7) \cdots (2l + 3)} A_0$$

$$A_{2l} = 0$$

for $l = 1, 2, 3, \cdots$, so that Eq. (53), (54), (55), and (56) also hold for $n = 1$.

Since the differential equations (46) are linear, their solution for $0 < x \leq 1$, when the initial conditions are

$$y_1 = a, \qquad y_2 = 0 \qquad\qquad \text{at } x = 1 \qquad (57)$$

is given by

$$y_1 = \alpha \, y_1^{\mathrm{I}}(x)$$

(58)

$$y_2 = \alpha \, y_2^{\mathrm{I}}(x)$$

where $y_1 = y_1^{\mathrm{I}}(x)$, $y_2 = y_2^{\mathrm{I}}(x)$ is the solution to Eq. (46) which is obtained by using the initial conditions

$$y_1 = 1, \; y_2 = 0 \qquad \qquad \text{at } x = 1$$

(59)

Because of boundary condition (47), we know that our initial conditions must be of the form of Eq. (57) and, assuming our solution is non-trivial, $\alpha \nless 0$.

Within the radius of convergence of the power series (55), it is required that the solutions match. Thus, corresponding to a given $\alpha \nless 0$, there is a value of A_0 such that

$$\alpha \, y_1^{\mathrm{I}}(h) = A_0 \, f_1(h)$$

(60)

$$\alpha \, y_2^{\mathrm{I}}(h) = A_0 \, f_2(h)$$

In order that a non-trivial solution exist, Eq. (60) necessitate that for a given non-negative integral value of n, the value of σ must make the following determinant vanish:

$$\begin{vmatrix} y_1^{\mathrm{I}}(h) & f_1(h) \\[2em] y_2^{\mathrm{I}}(h) & f_2(h) \end{vmatrix}$$

(61)

For each value of n there is a discrete set of values of σ which make the determinant (61) vanish. These sets of values might be referred to as the toroidal frequency spectrum of the structure, where the structure is defined once ρ_0 and μ have been assigned as functions of x and the value of a has been given. The values of σ which make the determinant vanish will occasionally be referred to as the eigenvalues.

Given a structure, the existing JPL IBM 7090 program is capable of generating monotonically the toroidal frequency spectrum for assigned n and the specified number of eigenvalues desired. The solution for y_1 and y_2 for each eigenvalue, corresponding to the initial conditions (59), is obtained for the interval

$$0 < h_2 \leq x \leq 1$$

where the value of h_2 is specified and where it is assumed that $h_2 \leq h$; the functions y_1 and y_2 may be automatically plotted if desired. As an example, with ρ and μ assumed constant, the first 20 eigenvalues for $n = 2$ required approximately 20 minutes of machine time for computation. For each eigenvalue, the approach is to converge in on a sequence of approximations for σ, halving the sum of the previous approximations which makes the determinant (61) change sign. The process is terminated when the values of the approximation are unchanged up to a specified number of significant figures.

A similar program is being planned to calculate the spheroidal frequency spectrum for an assumed structure. The approach being followed is similar to that used in the toroidal case, but the problem is more complicated.

IV. SOME ASPECTS OF THE SOLUTION FOR FREE SPHEROIDAL OSCILLATIONS

Introducing the variable x as defined by Eq. (45), the system of differential equations for the radial component of the spheroidal oscillations may be written as

$$\frac{dy_1}{dx} = -\frac{2\lambda}{(\lambda + 2\mu)} \frac{y_1}{x} + \frac{a}{(\lambda + 2\mu)} y_2 + \frac{\lambda n(n+1)}{(\lambda + 2\mu)} \frac{y_3}{x} \tag{62}$$

$$a \frac{dy_2}{dx} = \left[-\sigma^2 \rho_0 a^2 x^2 - 4\rho_0 g_0 a x + \frac{4\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} \right] \frac{y_1}{x^2} - \frac{4\mu a}{(\lambda + 2\mu)} \frac{y_2}{x}$$

$$+ \left[n(n+1) \rho_0 g_0 a x - \frac{2\mu(3\lambda + 2\mu)n(n+1)}{(\lambda + 2\mu)} \right] \frac{y_3}{x^2} + n(n+1) a \frac{y_4}{x} - \rho_0 a^2 y_6 \tag{63}$$

$$\frac{dy_3}{dx} = -\frac{y_1}{x} + \frac{y_3}{x} + \frac{a y_4}{\mu} \tag{64}$$

$$a \frac{dy_4}{dx} = \left[g_0 \rho_0 a x - \frac{2\mu(3\lambda + 2\mu)}{(\lambda + 2\mu)} \right] \frac{y_1}{x^2} - \frac{\lambda a}{(\lambda + 2\mu)} \frac{y_2}{x}$$

$$+ \left\{ -\rho_0 \sigma^2 a^2 x^2 + \frac{2\mu}{(\lambda + 2\mu)} [\lambda(2n^2 + 2n - 1) + 2\mu(n^2 + n - 1)] \right\} \frac{y_3}{x^2} - \frac{3a y_4}{x} - \frac{\rho_0 a^2 y_5}{ax} \tag{65}$$

$$\frac{1}{a} \frac{dy_5}{dx} = 4\pi G \rho_0 y_1 + y_6 \tag{66}$$

$$\frac{dy_6}{dx} = -4\pi G \rho_0 n(n+1) \frac{y_3}{x} + \frac{n(n+1)}{a} \frac{y_5}{x^2} - \frac{2y_6}{x} \tag{67}$$

where

$$g_0 = \frac{1}{x^2} \int_0^x 4\pi G \rho_0 a x^2 \, dx \qquad \text{for } x > 0$$

$$\tag{68}$$

$$= 0 \qquad \text{for } x = 0$$

The boundary conditions for the system are

$$y_2 = y_4 = 0 \quad , \quad y_6 + \frac{n+1}{a} y_5 = 0 \qquad \text{at } x = 1 \tag{69}$$

$$y_1, \ y_3, \text{ and } y_5 \text{ are regular} \qquad \text{at } x = 0 \tag{70}$$

To meet the conditions of regularity (Eq. 70), we assume that in the neighborhood of the origin

$$y_1 = x^r (A_0 + A_1 x + \cdots + A_k x^k + \cdots)$$

$$y_3 = x^s (B_0 + B_1 x + \cdots + B_k x^k + \cdots) \tag{71}$$

$$\frac{y_5}{a} = x^t (C_0 + C_1 x + \cdots + C_k x^k + \cdots)$$

where

$$A_0 \, B_0 \, C_0 \neq 0, \ r \geq 0, \ s \geq 0, \ t \geq 0$$

Using Eq. (62), (64), and (66) respectively to define $a y_2$, $a y_4$ and y_6 in terms of y_1, y_3, y_5/a and their first-order derivatives, we find on substituting Eq. (71) into Eq. (63), (65), and (67) that

$$r = s = n - 1 \quad , \quad t = n \tag{72}$$

The assumption that λ, μ, and ρ_0 are constant within a small neighborhood of the origin, say

$$\lambda = \overline{\lambda} \quad , \quad \mu = \overline{\mu} \quad , \quad \rho_0 = \overline{\rho} \qquad \text{for } x \leq h_1 \tag{73}$$

leads to

$$g_0 = \frac{4}{3} \pi G \overline{\rho} a x \qquad \text{for } x \leq h_1 \tag{74}$$

Substituting Eq. (71) into Eq. (63), (65), and (67) and making use of Eq. (72), (73), and (74) leads to the following set of equations which permit determination of A_k, B_k, and C_k as linear functions of A_0, B_2, and C_0:

$$B_0 = \frac{1}{n} A_0 \tag{75}$$

$$A_1 = B_1 = C_1 = 0 \tag{76}$$

$$[(n + 3) \overline{\lambda} + (n + 5) \overline{\mu}] A_2 = [(n^2 + n) \overline{\lambda} + (n^2 - n - 2) \overline{\mu}] B_2 \tag{77}$$

$$- \sigma^2 \overline{\rho} a^2 \frac{A_0}{n} + \frac{4}{3} \pi G \overline{\rho}^2 a^2 A_0 - \overline{\rho} a^2 C_0$$

$$A_{2l+1} = B_{2l+1} = C_{2l+1} = 0 \qquad \text{for } l = 1, 2, 3, \cdots \tag{78}$$

$$\{ [(n + 2l + 1)(n + 2l - 2)] \overline{\lambda} + [2(n + 2l)(n + 2l - 1) - n(n + 1) - 4] \overline{\mu} \} A_{2l}$$

$$- \{ [(n + 2l - 2) \overline{\lambda} + (n + 2l - 4) \overline{\mu}] \} n(n + 1) B_{2l}$$

$$= - \left(\frac{4}{3} \pi G \overline{\rho}^2 a^2 + \sigma^2 \overline{\rho} a^2 \right) A_{2l-2} + \left(\frac{4}{3} \pi G \overline{\rho}^2 a^2 \right) n(n + 1) B_{2l-2}$$

$$- \{ \overline{\rho} a^2 (n + 2l - 2) \} C_{2l-2} \tag{79}$$

$$\{ (n + 2l + 1) \overline{\lambda} + (n + 2l + 3) \overline{\mu} \} A_{2l} - \{ [n(n + 1)] \overline{\lambda} + [n(n + 1) - (2l - 1)(2n + 2l)] \overline{\mu} \} B_{2l}$$

$$= \left(\frac{4}{3} \pi G \overline{\rho}^2 a^2 \right) A_{2l-2} - (\overline{\rho} \sigma^2 a^2) B_{2l-2} - (\overline{\rho} a^2) C_{2l-2}$$

$$\text{for } l = 2, 3, \cdots$$

$$2l(2n + 2l + 1) C_{2l} = 4 \pi G \overline{\rho} [(n + 2l + 1) A_{2l} - n(n + 1) B_{2l}] \qquad \text{for } l = 1, 2, 3, \cdots \tag{80}$$

In a formal fashion, then, we may write for

$$x \leq h \leq h_1$$

$$y_1(x) = A_0 f_1(x) + B_2 g_1(x) + C_0 h_1(x)$$

$$y_2(x) = A_0 f_2(x) + B_2 g_2(x) + C_0 h_2(x)$$

$$y_3(x) = A_0 f_3(x) + B_2 g_3(x) + C_0 h_3(x)$$

$$y_4(x) = A_0 f_4(x) + B_2 g_4(x) + C_0 h_4(x) \tag{81}$$

$$y_5(x) = A_0 f_5(x) + B_2 g_5(x) + C_0 h_5(x)$$

$$y_6(x) = A_0 f_6(x) + B_2 g_6(x) + C_0 h_6(x)$$

where, for any specified value of $x \le h$ the 18 values of

$$f_j (x) \quad , \quad g_j (x) \quad , \quad h_j (x) \qquad\qquad (j = 1, 2, \cdots , 6)$$

are capable of being determined.

The most general solution to the system of differential equations (61) through (66) which satisfies the boundary conditions (69) is given by

$$y_1 = \alpha_1 y_1^I (x) + \alpha_2 y_1^{II} (x) + \alpha_3 y_1^{III} (x)$$

$$y_2 = \alpha_1 y_2^I (x) + \alpha_2 y_2^{II} (x) + \alpha_3 y_2^{III} (x)$$

$$y_3 = \alpha_1 y_3^I (x) + \alpha_2 y_3^{II} (x) + \alpha_3 y_3^{III} (x)$$

$$y_4 = \alpha_1 y_4^I (x) + \alpha_2 y_4^{II} (x) + \alpha_3 y_4^{III} (x)$$

$$y_5 = \alpha_1 y_5^I (x) + \alpha_2 y_5^{II} (x) + \alpha_3 y_5^{III} (x)$$

$$y_6 = \alpha_1 y_6^I (x) + \alpha_2 y_6^{II} (x) + \alpha_3 y_6^{III} (x)$$

$$(82)$$

where the functions $y_j^I (x)$, $y_j^{II} (x)$, $y_j^{III} (x)$; $j = 1, 2, \cdots , 6$, are the respective solutions to the system of differential equations when the values of $(y_1, y_2, y_3, y_4, y_5, y_6)$ at $x = 1$ are taken to be $(1, 0, 0, 0, 0, 0)$, $(0, 0, 1, 0, 0, 0)$, and $(0, 0, 0, 0, 1, - [n + 1]/a)$.

Imposing the condition that the solution to the system of differential equations must match the power series expansions at $x = h$, we obtain the eigenvalues for σ by requiring that the determinant

$$\begin{vmatrix} y_1^{\mathrm{I}} & y_1^{\mathrm{II}} & y_1^{\mathrm{III}} & f_1 & g_1 & h_1 \\ y_2^{\mathrm{I}} & y_2^{\mathrm{II}} & y_2^{\mathrm{III}} & f_2 & g_2 & h_2 \\ y_3^{\mathrm{I}} & y_3^{\mathrm{II}} & y_3^{\mathrm{III}} & f_3 & g_3 & h_3 \\ y_4^{\mathrm{I}} & y_4^{\mathrm{II}} & y_4^{\mathrm{III}} & f_4 & g_4 & h_4 \\ y_5^{\mathrm{I}} & y_5^{\mathrm{II}} & y_5^{\mathrm{III}} & f_5 & g_5 & h_5 \\ y_6^{\mathrm{I}} & y_6^{\mathrm{II}} & y_6^{\mathrm{III}} & f_6 & g_6 & h_6 \end{vmatrix}$$

must vanish at $x = h$.

Subsequently, the corresponding values of α_2/α_1 and α_3/α_1 can be determined and representative solutions $y_1, y_2, \cdots, y_6$ can be generated, at least in the interval ($h \leq x \leq 1$), by solving the system of differential equations with the initial conditions

$$\left(1, 0, \ \frac{\alpha_2}{\alpha_1}, 0, \ \frac{\alpha_3}{\alpha_1}, \ -\frac{n+1}{a} \ \frac{\alpha_3}{\alpha_1} \right)$$

V.   CONCLUDING REMARKS

A future report is planned to give more detail on the program for computing the spheroidal oscillations. If the numerical solution for the case of the liquid-core sphere is not forthcoming from those who have solved the problem, a future report will be planned to cover this problem.

NOMENCLATURE

a	radius of sphere before deformation
A_k	kth Frobenius coefficient
F	body forces
g_0	gravitational acceleration
G	gravitational constant
i, j, k	unit vectors in x, y, and z directions
P_0	initial pressure
P	defined by Eq. (34)
r	distance from center of sphere (polar coordinate system)
s	indicial constant
S_n	spherical surface harmonic
t	time
u_x, u_y, u_z	components of **u** in x, y, and z directions
u_r, u_θ, u_ϕ	components of **u** in r, θ, and ϕ directions
u	displacement vector
U	defined by Eq. (31)
$v(r)$	defined by Eq. (24)
$V(r)$	defined by Eq. (31)
W	perturbation of gravitational potential of sphere
W_s	gravitational potential outside sphere
x	dimensionless radius
X	defined by Eq. (33)
X, Y, Z	components of the body forces

NOMENCLATURE (Cont'd)

Δ defined by Eq. (4)

θ polar coordinate

λ elastic constant

μ rigidity

ρ density

ρ_0 initial density

σ frequency

ϕ polar coordinate

$\bar{\bar{\Phi}}$ symmetrical strain tensor

∇ gradient operator

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